Berry-Esseen and Edgeworth approximations for the tail of an infinite sum of weighted gamma random variables

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October 20, 2010

Abstract

Consider the sum $Z = \sum_{n=1}^{\infty} \lambda_n(\eta_n - \mathbb{E}\eta_n)$, where η_n are i.i.d. gamma random variables with shape parameter $r > 0$, and the λ_n's are predetermined weights. We study the asymptotic behavior of the tail $\sum_{n=M}^{\infty} \lambda_n(\eta_n - \mathbb{E}\eta_n)$ which is asymptotically normal under certain conditions. We derive a Berry-Essen bound and Edgeworth expansions for its distribution function. We illustrate the effectiveness of these expansions on an infinite sum of weighted chi-squared distributions.

1 Introduction

Consider a random variable given in terms of an infinite sum: $\sum_{n=1}^{\infty} \lambda_n(\eta_n - \mathbb{E}\eta_n)$, where η_n are i.i.d. gamma random variables with mean $\mathbb{E}\eta_n = r\theta$ and variance $\text{Var } \eta_n = r\theta^2$, where $r > 0$ and $\theta > 0$ are the shape and scale parameters, respectively. We may suppose without loss of generality that $\theta = r^{-1}$, by incorporating the extra parameter into the constants λ_n. We thus consider

$$Z = \sum_{n=1}^{\infty} \lambda_n(\eta_n - 1), \tag{1}$$

where η_n are i.i.d. gamma with pdf

$$f_{\eta_n}(x) = \frac{r^r}{\Gamma(r)} x^{r-1} e^{-rx}, \quad x > 0 \tag{2}$$

where $r > 0$. We suppose that $\{\lambda_n\}$ is a non-increasing sequence of positive numbers such that $\sum \lambda_n^2 < \infty$ and are normalized so that

$$\frac{1}{r} \sum \lambda_n^2 = 1. \tag{3}$$

With this setup, Z has mean zero and variance

$$\text{Var} Z = \sum_{n=1}^{\infty} \lambda_n^2 \text{Var}(\eta_n - 1) = \sum_{n=1}^{\infty} \frac{\lambda_n^2}{r} = 1. \tag{4}$$

Of particular interest is the case when $\lambda_n = n^{-\gamma} \ell(n)$, where $\gamma > 1/2$ and ℓ is slowly varying as $n \to \infty$. The restriction $\gamma > 1/2$ ensures $\sum \lambda_n^2 < \infty$ but allows for cases when either $\sum \lambda_n = \infty$ or $\sum \lambda_n < \infty$.

Random variables of the from (1) make up a rich class of distributions. Indeed, consider the double Weiner-Itô integral

$$I = \int_{\mathbb{R}^2}'' H(x,y)Z(dx)Z(dy) \tag{5}$$

where Z is a complex-valued Gaussian random measure. The double prime on the integral indicates that one excludes the diagonals $\{x = \pm y\}$ from the integration (for more on integrals of this type, see [12]). In [7], Proposition 2, Dobrushin and Major show that the random variable I can be expressed in the form (1) with $r = 1/2$ (chi-squared distributions). An important example in this case is the *Rosenblatt distribution*, discovered by M. Rosenblatt in [13], and later named after him in [18]. For an overview, see [17]. Properties of the Rosenblatt distribution are further developed in [19] using the results we obtain in the present paper.

A major difficulty that arises with distributions like (1) is that there is no closed form for its distribution function or density function. To make matters worse, even the characteristic function of Z is not easy to express or compute numerically. An initial approach to this problem might be to truncate the sum (1) at a level $M \geq 1$, and write $Z = X_M + Y_M$ where

$$X_M = \sum_{n=1}^{M-1} \lambda_n(\eta_n - 1), \qquad Y_M = \sum_{n=M}^{\infty} \lambda_n(\eta_n - 1) \tag{6}$$

and using X_M as an approximation of Z since it is a finite sum of weighted gamma distributions (an efficient method for computing the PDF/CDF of such a distribution can be found in [20]). How good is this approximation? This question can be partially answered by looking at the variance of Y_M,

$$\sigma_M^2 \equiv \mathrm{Var} Y_M = \frac{1}{r} \sum_{n=M}^{\infty} \lambda_n^2. \tag{7}$$

Depending on the decay of λ_n, this can tend to 0 slowly. For instance, if $\lambda_n \sim Cn^{-\gamma}$ for some $\gamma > 1/2$, then

$$\sigma_M^2 \sim \frac{C}{r} \int_M^{\infty} x^{-2\gamma} dx \sim \frac{C}{r} M^{1-2\gamma},$$

which tends to 0 slowly when γ is close to $1/2$, and thus in these cases M would have to be taken very large for X_M to be a reasonable approximation.

Instead of approximating Z by only X_M for M large, we will instead show when Y_M is asymptotically normal using a Berry-Essen estimate, and then we will give an Edgeworth expansion for the distribution function of Y_M. Combining this with the distribution of X_M will provide a method for computing the distribution function of Z. This fact can also be used for simulation of the random variable Z by simulating X_M exactly, and approximating the error with a $\mathrm{N}(0, \sigma_M^2)$ random variable.

This paper is organized as follows. In Section 2, we give the characteristic function of Z and Y_M in Lévy-Khintchine form. We then use this form of the characteristic function to show Y_M is asymptotically normal in Section 3. To approximate the CDF of Y_M, we prove an approximation lemma in Section 4 and in Section 5, we give an Edgeworth expansion. Finally, we demonstrate the accuracy of these approximations in Section 6 on an example where the η_n are chi-squared, and the sequence λ_n is given.

2 Lévy-Khintchine representation

Recall that a random variable X is infinitely divisible if for any positive integer n, one can find i.i.d. random variables $X_{1,n}, X_{2,n}, \ldots, X_{n,n}$ such that

$$X \stackrel{d}{=} X_{1,n} + X_{2,n} + \cdots + X_{n,n}$$

The characteristic function of any real valued infinitely divisible random variable X with $\mathbb{E}X^2 < \infty$ can be expressed in the following form, known as the Lévy-Khintchine form.

$$\mathbb{E}e^{iuX} = \exp\left(iau - \frac{1}{2}u^2\sigma^2 + \int_{\mathbb{R}\setminus\{0\}}(e^{iux} - 1 - iux)\Pi(dx)\right) \tag{8}$$

where $a \in \mathbb{R}$, $\sigma^2 > 0$ and Π is a measure on $\mathbb{R} \setminus \{0\}$, known as the *Lévy measure*, which satisfies

$$\int_{\mathbb{R}\setminus\{0\}}\min(x^2, 1)\Pi(dx) < \infty. \tag{9}$$

For background on such distributions see [16],[14], [4], or [2].

The random variable η with PDF (2) is infinitely divisible and has characteristic function

$$\mathbb{E}e^{iu\eta} = \exp\left(\int_0^\infty (e^{iux} - 1)\Pi(x)\right) \tag{10}$$

where the Lévy measure is given by $\Pi(dx) = rx^{-1}e^{-rx}dx$ for $x > 0$ ([2], example 1.3.22). Hence, if $\lambda > 0$, the random variable $\lambda(\eta - 1)$ is also infinitely divisible and its characteristic function is given by

$$\mathbb{E}\exp\left(iu\lambda(\eta - 1)\right) = \exp\left(\int_0^\infty (e^{iux} - 1 - iux)\left(\frac{r}{x}\exp\left(-\frac{rx}{\lambda}\right)\right)dx\right). \tag{11}$$

By taking an infinite sum of such distributions as in (1), it is not surprising that the resulting distribution is also infinitely divisible as indicated in the following proposition.

Proposition 2.1 *The characteristic function of Z defined in (1) is given by*

$$\mathbb{E}e^{iuZ} = \exp\left(\int_0^\infty (e^{iux} - 1 - iux)\nu(x)dx\right) \tag{12}$$

where ν is defined as

$$\nu(x) \equiv \frac{r}{x}\sum_{n=1}^\infty \exp\left(-\frac{rx}{\lambda_n}\right). \tag{13}$$

Proof. We have

$$\mathbb{E}e^{iuZ} = \lim_{M\to\infty}\mathbb{E}e^{iuX_M}$$

$$= \mathbb{E}\exp\left(\lim_{M\to\infty}\int_0^\infty (e^{iux} - 1 - iux)\left(\frac{r}{x}\sum_{n=1}^{M-1}\exp\left(-\frac{rx}{\lambda_n}\right)\right)dx\right). \tag{14}$$

To pass the limit through the integral above, note that $|e^{iux} - 1 - iux| \leq \frac{1}{2}u^2x^2$ and thus it suffices to show (using the dominated convergence theorem) that

$$\frac{r}{2}\int_0^\infty x\sum_{n=1}^\infty \exp\left(-\frac{rx}{\lambda_n}\right)dx < \infty. \tag{15}$$

This follows since

$$\int_0^\infty x\sum_{n=1}^\infty \exp\left(-\frac{rx}{\lambda_n}\right)dx = \sum_{n=1}^\infty \int_0^\infty x\exp\left(-\frac{rx}{\lambda_n}\right)dx = \sum_{n=1}^\infty \frac{\lambda_n^2}{r^2} < \infty.$$

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Thus (15) holds and hence the Lévy measure is given by (13).

∎

The form (12) of the characteristic function will be useful when we study the Edgeworth expansion of the tail Y_M defined in (6), whose Lévy measure is given by $\nu^{(M)}(x) = rx^{-1} \sum_{n=M}^{\infty} \exp\left(-\frac{rx}{\lambda_n}\right)$.

3   Berry-Esseen Bound

In this section we show that under certain conditions on the sequence λ_n, then the distribution of the tail Y_M is asymptotically normal as $M \to \infty$. A Berry-Esseen type bound for infinitely divisible random variables was studied in [3], and we will apply a similar method to the random variable Y_M.

Consider the normalized distribution $\widetilde{Y}_M = \sigma_M^{-1} Y_M$ where σ_M is defined in (7) and let

$$\widetilde{\nu}^{(M)}(x) = \sigma_M \nu^{(M)}(\sigma_M x) = \frac{r}{x} \sum_{n=M}^{\infty} \exp\left(-\frac{rx\sigma_M}{\lambda_n}\right), \quad x > 0 \tag{16}$$

be the density of the Lévy measure of $\widetilde{Y}_M$. As the remark below indicates, $\widetilde{Y}_M = \sigma_M^{-1} Y_M$ does not always converge to a normal distribution. To determine whether $\tilde{Y}_M$ is asymptotically normal, it suffices to consider the third cumulant of $\tilde{Y}_M$ which we denote by

$$\kappa_{3,M} \equiv \int_0^{\infty} x^3 \widetilde{\nu}^{(M)}(x)dx = r\sum_{n=M}^{\infty}\int_0^{\infty} x^2 e^{-rx\sigma_M/\lambda_n}dx = 2r^{-2}\sigma_M^{-3}\sum_{n=M}^{\infty}\lambda_n^3. \tag{17}$$

The following theorem uses a Berry-Esseen bound to show $\tilde{Y}_M$ is asymptotically normal if $\kappa_{3,M} \to 0$. The constant 0.7056 appearing in this bound is the smallest known to date, see [15].

Theorem 3.1 *Let Z be given by (1) and suppose the sequence λ_n is such that*

$$\frac{\displaystyle\sum_{n=M}^{\infty}\lambda_n^3}{\left(\displaystyle\sum_{n=M}^{\infty}\lambda_n^2\right)^{3/2}} \longrightarrow 0 \quad as\ M \to \infty. \tag{18}$$

Then, $\tilde{Y}_M \to \mathrm{N}(0,1)$ as $M \to \infty$ and we have

$$\sup_{x\in\mathbb{R}}\left|\mathrm{P}[\tilde{Y}_M \le x] - \Phi(x)\right| \le 0.7056\ \kappa_{3,M} \tag{19}$$

where Φ is the standard normal CDF and $\kappa_{3,M}$ is defined in (17).

Remark: It can easily be checked that condition (18) is satisfied if λ_n decays as a power law, i.e. if $\lambda_n \sim Cn^{-\gamma}$ for some $\gamma > 1/2$. However (18) is *not* satisfied if λ_n decays exponentially, and in this case convergence to $\mathrm{N}(0,1)$ will not always hold. For example, suppose $\lambda_n = 2^{-n-1}$. Then $Y_M = \sum \lambda_n(\eta_n - 1) = \sum \lambda_n \eta_n - \sum \lambda_n \ge -\sum \lambda_n$ and so

$$\sigma_M^{-1} Y_M \ge \left(r^{-1}\sum_{n=M}^{\infty}\lambda_n^2\right)^{-1/2}\left(-\sum_{n=M}^{\infty}\lambda_n\right) = -\left(\sum_{n=M}^{\infty}2^{-n-1}\right)\Bigg/\sqrt{r^{-1}\sum_{n=M}^{\infty}2^{-2n-2}} = -\sqrt{3r}$$

for all M. Since the normalized random variable $\sigma_M^{-1} Y_M$ is bounded below, it cannot converge in distribution to $\mathrm{N}(0,1)$.

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Proof. Since Y_M is infinitely divisible, for each $n \geq 1$ we have

$$Y_M = Y_{M,1}^{(n)} + Y_{M,2}^{(n)} + \cdots + Y_{M,n}^{(n)}, \tag{20}$$

where $Y_{M,i}^{(n)}$, $i = 1, 2, \ldots, n$ are i.i.d. with mean 0 and variance σ_M^2/n. Applying the Berry-Esseen Theorem ([8], Theorem 7.6.1) to the sum (20), we have for any $n \geq 1$,

$$
\begin{aligned}
\sup_{x \in \mathbb{R}} \left| \mathrm{P}\left[\sigma_M^{-1} Y_M \leq x\right] - \Phi(x) \right| &= \sup_{x \in \mathbb{R}} \left| \mathrm{P}\left[\frac{1}{\sigma_M \sqrt{n}}(\sqrt{n} Y_M) \leq x\right] - \Phi(x) \right| \\
&\leq 0.7056 \frac{\mathbb{E}\left[\sqrt{n} Y_{M,1}^{(n)}\right]^3}{\sigma_M^3 \sqrt{n}} \\
&= 0.7056 \frac{n\mathbb{E}[Y_{M,1}^{(n)}]^3}{\sigma_M^3}.
\end{aligned} \tag{21}
$$

Using Lemma 3.1 in [3], $n\mathbb{E}[\sigma_M^{-1} Y_{M,1}^{(n)}]^3 \to \int_0^\infty x^3 \tilde{\nu}_D^{(M)}(x)dx = \kappa_{3,M}$. Thus, we let $n \to \infty$ in (21), which gives (19).

To see that the right hand side of this bound tends to 0 as $M \to \infty$, notice that by (7),

$$\kappa_{3,M} = 2r^{-2}\sigma_M^{-3} \sum_{n=M}^\infty \lambda_n^3 = 2r^{-2}\left(\frac{1}{r}\sum_{n=M}^\infty \lambda_n^2\right)^{-3/2} \sum_{n=M}^\infty \lambda_n^3, \tag{22}$$

which tends to 0 by the assumption (18), implying convergence to $\mathrm{N}(0,1)$. This finishes the proof. ∎

If $\lambda_n \sim Cn^{-\gamma}$ for $\gamma > 1/2$, then $\kappa_{3,M} = O(M^{-1/2})$ (see (26) below), which describes the rate at which the right hand side of (19) tends to 0. While it is nice to have a practical bound on the error made when approximating the CDF of Y_M with that of a normal, this rate of convergence may be too slow. In the next section, we improve this approximation by using Edgeworth expansions. These will do a better job of approximating the CDF of Y_M for small M, however it will no longer be easy to bound the error made in this approximation exactly.

4 An approximation lemma

The previous section showed that the tail Y_M can be approximated by a normal distribution for large M. We shall improve the approximation to the CDF of Y_M using an Edgeworth expansion. To establish the Edgeworth expansion, we will need a lemma involving an approximation of the characteristic function of $\tilde{Y}_M$ by a polynomial involving the cumulants.

In the following sections, we will make the following assumption about the sequence λ_n:

$$\lambda_n = \ell(n)n^{-\gamma} \tag{23}$$

where $\gamma > 1/2$, and ℓ is a slowly varying function at ∞. With this assumption, (3) is satisfied and

$$\sigma_M^2 \sim \frac{1}{r}\int_M^\infty \ell(n)^2 n^{-2\gamma}dn = \frac{\ell(M)^2 M^{1-2\gamma}}{r(1-2\gamma)}. \tag{24}$$

Extending the definition of $\kappa_{3,M}$ in (17), we will denote all cumulants of $\tilde{Y}_M$ by (see [16], Theorem 7.4),

$$
\begin{aligned}
\kappa_{k,M} = \int_0^\infty x^k \tilde{\nu}^{(M)}(x)dx &= r \sum_{n=M}^\infty \int_0^\infty x^{k-1} \exp\left(-\frac{rx\sigma_M}{\lambda_n}\right)dx \\
&= \frac{(k-1)!}{r^{k-1}\sigma_M^k} \sum_{n=M}^\infty \lambda_n^k, \quad k \geq 2.
\end{aligned}
\tag{25}
$$

Observe that $\kappa_{2,M} = 1$ and as $M \to \infty$, (23), (24), (25) and properties of slowly varying functions imply

$$
\begin{aligned}
\kappa_{k,M} \sim \frac{(k-1)!}{r^{k-1}}\sigma_M^{-k} \int_M^\infty \ell(n)^k n^{-k\gamma}dn &\sim C_k(\ell(M)^{-k}M^{-(\frac{k}{2}-k\gamma)})(\ell(M)^k M^{1-k\gamma}) \\
&= C_k M^{1-\frac{k}{2}}, \quad k \geq 2
\end{aligned}
\tag{26}
$$

for a constant C_k. Notice that in particular, if $k = 3$, then $\kappa_{3,M} \sim C_3 M^{-1/2}$, which implies condition (18).

In view of Proposition 2.1, the difference between the log of the characteristic function of $\tilde{Y}_M$ and that of a standard normal is given by the following function I_M defined as

$$
I_M(u) \equiv \int_0^\infty \left(e^{iux} - 1 - iux - \frac{(iux)^2}{2}\right)\tilde{\nu}^{(M)}(x)dx,
\tag{27}
$$

which can be rewritten as

$$
I_M(u) = \int_0^\infty (e^{iux} - 1 - iux)\tilde{\nu}^{(M)}(x)dx - \left(-\frac{u^2}{2}\right),
\tag{28}
$$

since $\int_0^\infty x^2 \tilde{\nu}^{(M)}(x)dx = \kappa_{2,M} = 1$. A key step in developing an Edgeworth expansion is approximating the function $e^{I_M(u)}$ by a polynomial involving the cumulants, which is done in the following lemma.

Lemma 4.1 *For $N \geq 3$ and $u > 0$, we have as $M \to \infty$,*

$$
\left| e^{I_M(u)} - \left[1 + \sum_{\eta(N)} \left(\prod_{m=3}^N \frac{1}{k_m!}\left(\frac{(iu)^m}{m!}\kappa_{m,M}\right)^{k_m}\right)\right] \right| \leq Q_N(u) + \frac{u^{3N-3}}{(3!)^{N-1}(N-1)!}\kappa_{3,M}^{N-1} \exp\left(\frac{u^3}{6}\kappa_{3,M}\right),
\tag{29}
$$

where $\eta(N)$ denotes all non-negative indices $k_3, k_4, \ldots, k_N$ such that

$$
1 \leq k_3 + 2k_4 + \ldots (N-2)k_N \leq N - 2
\tag{30}
$$

and $|Q_N(u)|$ is bounded by a polynomial in u whose coefficients are $O\left(M^{-\left(\frac{N-1}{2}\right)}\right)$ as $M \to \infty$.

Remark: This bound is a complicated function of u, but this will cause no problem because in the proof of Theorem 5.2 below, this bound is multiplied by $e^{-u^2/2}$ and integrated over $u \in [0, \kappa_{3,M}^{-1}]$.

Proof. By using Taylor's Theorem on the function e^{iux} for $u \geq 0$, we have for each $N \geq 2$,

$$
I_M(u) = \int_0^\infty \left(e^{iux} - 1 - iux - \frac{(iux)^2}{2}\right)\tilde{\nu}^{(M)}(dx) = \int_0^\infty \left(\sum_{m=3}^N \frac{(iux)^m}{m!} + R_N(ux)\right)\tilde{\nu}^{(M)}(dx)
\tag{31}
$$

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where R_N is a remainder which satisfies

$$|R_N(ux)| \leq \frac{(ux)^{N+1}}{(N+1)!}.$$

Using the definition (25) of $\kappa_{k,M}$, I_M becomes

$$I_M(u) = \sum_{m=3}^{N} \frac{(iu)^m}{m!} \kappa_{m,M} + \widetilde{R}_N(u) \tag{32}$$

where now,

$$|\widetilde{R}_N(u)| \leq \int_0^\infty \frac{(ux)^{N+1}}{(N+1)!} \tilde{\nu}^{(M)}(x) dx = \frac{u^{N+1}}{(N+1)!} \kappa_{N+1,M}. \tag{33}$$

Notice that $I_M(u) = \widetilde{R}_2(u)$, which follows from (33) by setting $N = 2$.

Turning now to $\exp(I_M(u))$, we apply the classical inequality

$$\left| e^z - \sum_{n=0}^{r} \frac{z^n}{n!} \right| \leq \frac{z^{r+1}}{r!} e^{|z|}, \quad z \in \mathbb{R}, \ r \geq 0 \tag{34}$$

to $\exp(I_M(u))$ and using (33), we get

$$\begin{aligned}
\left| \exp(I_M(u)) - \sum_{n=0}^{N-2} \frac{I_M(u)^n}{n!} \right| &\leq \frac{|I_M(u)|^{N-1}}{(N-1)!} \exp(|I_M(u)|) \\
&= \frac{|\widetilde{R}_2(u)|^{N-1}}{(N-1)!} \exp(|\widetilde{R}_2(u)|) \\
&\leq \frac{u^{3(N-1)}}{(3!)^{N-1}(N-1)!} \kappa_{3,M}^{N-1} \exp\left(\frac{u^3}{3!} \kappa_{3,M} \right).
\end{aligned} \tag{35}$$

Thus, by adding and subtracting $\sum_{n=1}^{N-2} \frac{I_M(u)^n}{n!}$ on the left hand side of (29), we have

$$\begin{aligned}
&\left| \exp(I_M(u)) - \left(1 + \left[\sum_{\eta(N)} \left(\prod_{m=3}^{N} \frac{1}{k_m!} \left(\frac{(iu)^m}{m!} \kappa_{m,M} \right)^{k_m} \right) \right] \right) \right| \\
&\leq \left| \exp(I_M(u)) - \sum_{n=0}^{N-2} \frac{I_M(u)^n}{n!} \right| \\
&+ \left| \sum_{n=0}^{N-2} \frac{I_M(u)^n}{n!} - \left(\left[\sum_{\eta(N)} \left(\prod_{m=3}^{N} \frac{1}{k_m!} \left(\frac{(iu)^m}{m!} \kappa_{m,M} \right)^{k_m} \right) \right] \right) \right|.
\end{aligned} \tag{36}$$

Notice (35) gives a bound for the first term in (36). Thus, to finish the proof it remains to bound the second term in (36). To do this, fix $1 \leq n \leq N-2$ and observe that (32) implies

$$\frac{I_M(u)^n}{n!} = \frac{1}{n!} \left(\sum_{m=3}^{N} \frac{(iu)^m}{m!} \kappa_{m,M} + \widetilde{R}_N(u) \right)^n \tag{37}$$

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Applying the multinomial theorem, this becomes

$$
\begin{aligned}
\frac{I_M(u)^n}{n!} &= \frac{1}{n!}\sum_{\{k_m\}_n}\binom{n}{k_3,k_4,\ldots k_N,k_{N+1}}\left[\prod_{m=3}^{N}\left(\frac{(iu)^m}{m!}\kappa_{m,M}\right)^{k_m}\right]\widetilde{R}_N^{k_{N+1}} \\
&= \sum_{\{k_m\}_n}\left[\prod_{m=3}^{N}\frac{1}{k_m!}\left(\frac{(iu)^m}{m!}\kappa_{m,M}\right)^{k_m}\right]\frac{\widetilde{R}_N^{k_{N+1}}}{k_{N+1}!}
\end{aligned}
\tag{38}
$$

where $\{k_m\}_n$ denotes all sets of non-negative integers k_m, $3 \leq m \leq N+1$ such that $k_3+k_4+\cdots+k_N+k_{N+1} = n$. By (26), $\kappa_{m,M} = O(1)$. Moreover, by (33) and (26), $|\widetilde{R}_N| \leq \frac{u^{N+1}}{(N+1)!}\kappa_{N+1,M} \sim \frac{u^{N+1}}{(N+1)!}C_{N+1}M^{-(N-1)/2}$, thus any term in (38) involving $\tilde{R}_N$ (that is with $k_{N+1} \geq 1$) can be grouped into a function $Q_{n,N}^{(1)}(u)$ which is bounded by a polynomial with positive coefficients which are $O(M^{-(N-1)/2})$. Doing this, (38) becomes

$$
\frac{I_M(u)^n}{n!} = \sum_{\{k_m\}_n'}\left[\prod_{m=3}^{N}\frac{1}{k_m!}\left(\frac{(iu)^m}{m!}\kappa_{m,M}\right)^{k_m}\right] + Q_{n,N}^{(1)}(u)
\tag{39}
$$

where $\{k_m\}_n'$ denotes all k_m, $3 \leq m \leq N$ such that $k_3 + k_4 + \ldots k_N = n$. In the remaining sum, the coefficients are

$$
\prod_{m=3}^{N}\frac{1}{k_m!(m!)^{k_m}}\kappa_{m,M}^{k_m}.
\tag{40}
$$

Using (26) again, these coefficients are of the order

$$
\begin{aligned}
\prod_{m=1}^{N}\kappa_{m,M}^{k_m} &= O\left(M^{-\sum_{m=3}^{N}k_m\left(\frac{m-2}{2}\right)}\right) \\
&= O\left(M^{-\frac{1}{2}\left[\sum_{m=3}^{N}mk_m-2\sum_{m=3}^{N}k_m\right]}\right) \\
&= O\left(M^{-\frac{1}{2}\left[\sum_{m=3}^{N}mk_m-2n\right]}\right).
\end{aligned}
\tag{41}
$$

We shall now isolate the terms in the sum (39) for which

$$
\sum_{m=3}^{N}mk_m \geq N + 2n - 1.
\tag{42}
$$

They form a polynomial $Q_{n,N}^{(2)}(u)$ whose coefficients by (41) are of the order

$$
O\left(M^{-\frac{1}{2}\left[\sum_{m=3}^{N}mk_m-2n\right]}\right) = O\left(M^{-\frac{1}{2}[N+2n-1-2n]}\right) = O\left(M^{-(N-1)/2}\right),
$$

where we have used the fact that the k_m's are chosen to satisfy (42). Thus,

$$
\frac{I_M(u)^n}{n!} = \sum_{\{k_m\}_n''}\left[\prod_{m=3}^{N}\frac{1}{k_m!}\left(\frac{(iu)^m}{m!}\kappa_{m,M}\right)^{k_m}\right] + Q_{n,N}^{(1)}(u) + Q_{n,N}^{(2)}(u)
\tag{43}
$$

where $\{k_m\}_n''$ denotes all k_m, $3 \leq m \leq N$ such that $\sum k_m = n$ and $\sum mk_m \leq N + 2n - 2$. Notice that by combining these two inequalities, the k_m's in this sum also satisfy

$$
\sum_{m=3}^{N}(m-2)k_m \leq (N+2n-2) - 2n = N - 2.
\tag{44}
$$

Now, returning to the second term in (36), in light of (43), we have

$$\sum_{n=1}^{N-2} \frac{I_M(u)^n}{n!} = \left(\sum_{n=1}^{N-2} \left[\sum_{\{k_m\}_n''} \left(\prod_{m=3}^{n} \frac{1}{k_m!} \left(\frac{(iu)^m}{m!} \kappa_{m,M} \right)^{k_m} \right) \right] \right) + \sum_{n=1}^{N-2} [Q_{n,N}^{(1)}(u) + Q_{n,N}^{(2)}(u)]$$

$$= \left(\sum_{n=1}^{N-2} \left[\sum_{\{k_m\}_n''} \left(\prod_{m=3}^{n} \frac{1}{k_m!} \left(\frac{(iu)^m}{m!} \kappa_{m,M} \right)^{k_m} \right) \right] \right) + Q_N(u) \qquad (45)$$

where $Q_N(u) := \sum_{n=1}^{N-2} [Q_{n,N}^{(1)}(u) + Q_{n,N}^{(2)}(u)]$ is bounded by a polynomial in u whose coefficients are $O(M^{-(\frac{N-1}{2})})$. As for the double sum on the right hand side of (45), observe that from (44), this can be rewritten as the (single) sum over all k_i, $3 \le i \le N$ such that

$$1 \le k_3 + k_4 + \dots k_N \le N - 2 \quad \text{and}$$
$$k_3 + 2k_4 + \dots + (N-2)k_N \le N - 2.$$

Since $k_3 + k_4 + \dots + k_N \le k_3 + 2k_4 + \dots (N-2)k_N$, these two conditions are satisfied if and only if

$$1 \le k_3 + 2k_4 + \dots + (N-2)k_N \le N - 2,$$

which is the definition of $\eta(N)$ in (30). Thus,

$$\left| \sum_{n=1}^{N-2} \frac{I_M(u)^n}{n!} - \sum_{\eta(N)} \prod_{m=3}^{N} \frac{1}{k_m!} \left(\frac{(iu)^m}{m!} \kappa_{m,M} \right)^{k_m} \right| \le |Q_N(u)|. \qquad (46)$$

This bounds the second term in (36) and completes the proof.

∎

5 Edgeworth expansions

We shall improve the approximation to the CDF of Y_M using and Edgeworth expansion. A two-term Edgeworth expansion of a general sequence of infinitely divisible distributions are studied in [10]. We apply a similar method to our case, but with an Edgewood expansion to any order.

Given a CDF F of a random variable X and a function G (not necessarily a CDF) we let ρ denote the supremum norm of the difference $F - G$:

$$\rho(F, G) = \sup_{x \in \mathbb{R}} |F(x) - G(x)|$$

We can bound $\rho(F, G)$ using the characteristic function of X and the Fourier-Stieltjes transform of G. This is done in the following lemma which is proved in [5], Lemma 12.2.

Lemma 5.1 *Let ϕ be a characteristic function of a random variable X with CDF F. Let G be a function for which*

$$\lim_{x \to -\infty} G(x) = 0, \quad \lim_{x \to \infty} G(x) = 1, \quad and \quad \sup_{x \in \mathbb{R}} |G'(x)| < C,$$

for some constant C and let $g(u) = \int_{\mathbb{R}} e^{iux} dG(x)$ be the Fourier-Stieltjes transform of G. Furthermore, suppose that

$$\int_{\mathbb{R}} |x| dF(x) < \infty \quad and \quad \int_{\mathbb{R}} |x| dG(x) < \infty.$$

Then for every $U > 0$ and $t > t_0$,

$$\rho(F, G) \leq \frac{1}{4h(t) - \pi} \int_0^U |\phi(u) - g(u)| \frac{du}{u} + 4th(t)\frac{C}{U} \tag{47}$$

where h and t_0 are defined as

$$h(t) = \int_0^t \frac{\sin^2(x)}{x^2} dx, \ t > 0, \quad and \quad h(t_0) = \frac{\pi}{4}.$$

This lemma involves two parameters t and U, which must balance each other (making U large decreases the second term on the right hand side of (47) and increases the first, and t has the opposite effect). In our application, U will tend to infinity and t will be an unspecified constant. This lemma will be used to study the convergence of an Edgewood expansion for $\tilde{Y}_M$.

We can now state a theorem detailing the convergence rate of an Edgeworth expansion for the CDF of $\tilde{Y}_M$ as $M \to \infty$. Recall the *Hermite polynomials* which can be defined as $H_0(x) = 1$ and

$$H_k(x) = (-1)^k e^{x^2/2} \frac{d^k}{dx^k} e^{-x^2/2}, \quad k \geq 1,$$

see [9], page 157. A simple induction shows that H_k also satisfies the recursion formula

$$H_{k+1}(x) = -e^{x^2/2} \frac{d}{dx} \left(H_k(x) e^{-x^2/2} \right), \quad k \geq 0. \tag{48}$$

The first few H_k are given by $H_1(x) = x$, $H_2(x) = x^2 - 1$, $H_3(x) = x^3 - 3x$, $H_4(x) = x^4 - 6x^2 + 3$, $H_5(x) = x^5 - 10x^3 + 15x, \ldots$.

The following theorem provides an Edgeworth expansion of $\tilde{Y}_M$ up to an arbitrary order $N \geq 2$.

Theorem 5.2 *As $M \to \infty$, for each $N \geq 2$ the CDF of $\tilde{Y}_M$ satisfies*

$$P[\tilde{Y}_M \leq x] = \Phi(x) - \phi(x) \left\{ \sum_{\eta(N)} \left[\prod_{m=1}^N \frac{1}{k_m!} \left(\frac{\kappa_{m,M}}{m!} \right)^{k_m} \right] H_{\zeta(k_3,\ldots,k_N)}(x) \right\} + O\left(M^{-\frac{N-1}{2}} \right), \tag{49}$$

where Φ and ϕ denote the standard normal CDF and PDF, $\eta(N)$ denotes all non-negative indices $k_3, k_4, \ldots, k_N$ such that

$$1 \leq k_3 + 2k_4 + \ldots (N-2)k_N \leq N - 2 \tag{50}$$

and

$$\zeta(k_1, \ldots, k_N) = 3k_3 + 4k_4 + \ldots Nk_N - 1. \tag{51}$$

Moreover, the error $O(M^{-(N-1)/2})$ is uniform for all $x \in \mathbb{R}$.

For example, if $N = 2$, there is no solution to (50). If $N = 3$, the only solution to (50) is $k_3 = 1$. If $N = 4$, we have the additional solutions $k_3 = 2$, $k_4 = 0$ and $k_3 = 0$, $k_4 = 1$. Thus, for small values of N, the right hand side of (49) becomes

$$\mathbf{N = 2}: \quad \mathrm{P}[\tilde{Y}_M \leq x] = \Phi(x) + O(M^{-1/2})$$

$$\mathbf{N = 3}: \quad \mathrm{P}[\tilde{Y}_M \leq x] = \Phi(x) - \phi(x)\frac{H_2(x)}{3!}\kappa_{3,M} + O(M^{-1})$$

$$\mathbf{N = 4}: \quad \mathrm{P}[\tilde{Y}_M \leq x] = \Phi(x) - \phi(x)\left[\frac{H_2(x)}{3!}\kappa_{3,M} + \left(\frac{H_3(x)}{4!}\kappa_{4,M} + \frac{H_5(x)}{(2!)(3!)^2}\kappa_{3,M}^2\right)\right] + O(M^{-3/2})$$

$$\mathbf{N = 5}: \quad \mathrm{P}[\tilde{Y}_M \leq x] = \Phi(x) - \phi(x)\left[\frac{H_2(x)}{3!}\kappa_{3,M} + \left(\frac{H_3(x)}{4!}\kappa_{4,M} + \frac{H_5(x)}{(2!)(3!)^2}\kappa_{3,M}^2\right)\right.$$
$$\left. + \left(\frac{H_4(x)}{5!}\kappa_{5,M} + \frac{H_6(x)}{(3!)(4!)}\kappa_{4,M}\kappa_{3,M} + \frac{H_8(x)}{(3!)^4}\kappa_{3,M}^3\right)\right] + O(M^{-2})$$

A more revealing (but slightly more complicated) statement of Theorem 5.2 is

$$\mathrm{P}[\tilde{Y}_M \leq x] = \Phi(x) - \phi(x)\left\{\sum_{n=3}^{N}\left(\sum_{\eta'(n)}\left[\prod_{m=3}^{n}\frac{1}{k_m!}\left(\frac{\kappa_{m,M}}{m!}\right)^{k_m}\right]H_{\zeta(k_3,\dots,k_n)}(x)\right)\right\} + O\left(M^{-\frac{N-1}{2}}\right), \quad (52)$$

where $\eta'(n)$ denotes all $k_3, k_4, \dots, k_N$ such that $k_3 + 2k_4 + \dots + (n-2)k_N = n - 2$. In this form, it is clearer what additional terms appear in the expansion as you increase n from 3 to N.

Proof of Theorem 5.2

Proof.   Define $G(x)$ as

$$G(x) = \Phi(x) - \phi(x)\left\{\sum_{\eta(N)}\left[\prod_{m=1}^{N}\frac{1}{k_m!}\left(\frac{\kappa_{m,M}}{m!}\right)^{k_m}\right]H_{\zeta(k_3,\dots,k_N)}(x)\right\}. \quad (53)$$

Then by (48), we also have

$$\frac{dG}{dx} = \phi(x)\left(1 + \sum_{\eta(N)}\left[\prod_{m=1}^{N}\frac{1}{k_m!}\left(\frac{\kappa_{m,M}}{m!}\right)^{k_m}\right]H_{\zeta(k_3,\dots,k_N)+1}\right).$$

Using this and the fact $\int_{\mathbb{R}} H_k(x)\phi(x)e^{iux}\,dx = (-1)^k\int_{\mathbb{R}}\left(\frac{d^k}{dx^k}\phi(x)\right)e^{iux}\,dx = (iu)^k e^{-u^2/2}$, the Fourier-Stieltjes transform of G is given by

$$g(u) = \int_{\mathbb{R}}e^{iux}\,dG(x) \;=\; \exp\left(-\frac{u^2}{2}\right)\left(1 + \sum_{\eta(N)}\left(\prod_{m=3}^{N}\frac{1}{k_m!}\left(\frac{\kappa_{m,M}}{m!}\right)^{k_m}(iu)^{\zeta(k_3,\dots,k_N)+1}\right)\right)$$

$$\;=\; \exp\left(-\frac{u^2}{2}\right)\left(1 + \sum_{\eta(N)}\left(\prod_{m=3}^{N}\frac{1}{k_m!}\left(\frac{(iu)^m}{m!}\kappa_{m,M}\right)^{k_m}\right)\right) \quad (54)$$

where we have used the definition of ζ in (51). Let $\varphi^{(M)}(u)$ be the characteristic function of $\tilde{Y}_M$:

$$\varphi^{(M)}(u) = \exp\left(\int_0^{\infty}(e^{iux} - 1 - iux)\tilde{\nu}^{(M)}(x)\,dx\right). \quad (55)$$

Since $N \geq 2$, choose $\epsilon > 0$ such that
$$\kappa_{3,M}^{-1} < \epsilon \kappa_{N+1,M}^{-1}$$
for all $M \geq 1$ (this exists by (26)). To show (49) using Lemma 5.1, it suffices to show that

$$J_M := \int_0^{U_M} |\varphi^{(M)}(u) - g(u)| \frac{du}{u} = O\left(M^{-(N-1)/2}\right), \tag{56}$$

where

$$U_M := \epsilon \kappa_{N+1,M}^{-1} \sim C_{N+1} M^{-\left(1 - \frac{N+1}{2}\right)} = C_{N+1} M^{\frac{N-1}{2}} \tag{57}$$

from (26). Notice that with this choice of U_M, the second term on the right hand side of (47) is already of order $O(U_M^{-1}) = O(M^{-(N-1)/2})$ and thus we need to only bound J_M.

Using (28), notice that

$$\varphi^{(M)}(u) = \exp\left(\int_0^\infty (e^{iux} - 1 - iux)\widetilde{\nu}^{(M)}(x)dx\right) = \exp\left(I_M(u) - \frac{u^2}{2}\right).$$

Using this and the definition of g in (54), we can break up the integral J_M in (56) as

$$
\begin{aligned}
J_M &= \int_0^{U_M} \exp\left(-\frac{u^2}{2}\right)\left(\exp(I_M(u)) - \left(1 + \sum_{\eta(N)}\left(\prod_{m=3}^N \frac{1}{k_m!}\left(\frac{(iu)^m}{m!}\kappa_{m,M}\right)^{k_m}\right)\right)\right)\frac{du}{u} \\
&:= J_{M,1} + J_{M,2} + J_{M,3},
\end{aligned}
$$

where

$$J_{M,1} = \int_0^{\kappa_{3,M}^{-1}} \exp\left(-\frac{u^2}{2}\right)\left(\exp(I_M(u)) - \left(1 + \sum_{\eta(N)}\left(\prod_{m=3}^N \frac{1}{k_m!}\left(\frac{(iu)^m}{m!}\kappa_{m,M}\right)^{k_m}\right)\right)\right)\frac{du}{u}$$

$$J_{M,2} = \int_{\kappa_{3,M}^{-1}}^{U_M} \exp\left(-\frac{u^2}{2} + I_M(u)\right)\frac{du}{u}$$

$$J_{M,3} = -\int_{\kappa_{3,M}^{-1}}^{U_M} \exp\left(-\frac{u^2}{2}\right)\left(1 + \sum_{\eta(N)}\left(\prod_{m=3}^N \frac{1}{k_m!}\left(\frac{(iu)^m}{m!}\kappa_{m,M}\right)^{k_m}\right)\right)\frac{du}{u}.$$

We will now show that $J_{M,i} = O(M^{-(N-1)/2})$, $i = 1, 2, 3$, which with the help of Lemma 5.1, will imply the result.

Estimate for $J_{M,1}$:

From Lemma 4.1, we have that

$$|J_{M,1}| \leq \int_0^{\kappa_{3,M}^{-1}} \exp\left(-\frac{u^2}{2}\right)\left|Q_N(u) + \frac{u^{3N-3}}{(3!)^{N-1}(N-1)!}\kappa_{3,M}^{N-1}\exp\left(\frac{u^3}{6}\kappa_{3,M}\right)\right|\frac{du}{u}$$

where Q_N is bounded by a polynomial in u whose coefficients are $O(M^{-(N-1)/2})$. Thus,

$$|J_{M,1}| \leq \int_0^{\kappa_{3,M}^{-1}} \exp\left(-\frac{u^2}{2}\right)|Q_N(u)|\frac{du}{u} + \int_0^{\kappa_{3,M}^{-1}} \exp\left(-\frac{u^2}{3}\right)\frac{u^{3N-4}}{(3!)^{N-1}(N-1)!}\kappa_{3,m}^{N-1}du, \tag{58}$$

since on the interval $0 < u < \kappa_{3,M}^{-1}$, we have $u\kappa_{3,M} \le 1$ and

$$\exp\left(-\frac{u^2}{2}\right)\exp\left(\frac{u^3}{6}\kappa_{3,M}\right) \le \exp\left(-\frac{u^2}{3}\right).$$

The first term in (58) is $O(M^{-(N-1)/2})$ since all coefficients of Q_N are of this order. The second term in (58) is also of order $O(M^{-(N-1)/2})$ by (26). Thus, $J_{M,1} = O(M^{-(N-1)/2})$.

Estimate for $J_{M,2}$:

We will in fact show that $J_{2,M} = o(M^{-(N-1)/2})$. First, observe that

$$|J_{2,M}| \le \int_{\kappa_{3,M}^{-1}}^{U_M} \exp\left(-\frac{u^2}{2} + \mathrm{Re}[I_M(u)]\right)\frac{du}{u}. \tag{59}$$

Thus, we must show the integrand tends to zero fast enough. Notice, using (28), that

$$-\frac{u^2}{2} + \mathrm{Re}[I_M(u)] = -\int_0^\infty (1-\cos(ux))\widetilde{\nu}^{(M)}(x)dx$$

Using (16), we compute this integral:

$$
\begin{aligned}
A_M(u) &:= \int_0^\infty (1-\cos(ux))\widetilde{\nu}^{(M)}(x)dx = r\int_0^\infty (1-\cos(ux))\left(\sum_{n=M}^\infty \frac{e^{-xr\sigma_M/\lambda_n}}{x}\right)dx \\
&= r\sum_{n=M}^\infty \int_0^\infty \frac{1-\cos(ux)}{x}e^{-xr\sigma_M/\lambda_n}dx = r\sum_{n=M}^\infty \int_0^u \int_0^\infty \sin(tx)e^{-xr\sigma_M/\lambda_n}dtdx \\
&= r\sum_{n=M}^\infty \int_0^u \left(\int_0^\infty \sin(tx)e^{-xr\sigma_M/\lambda_n}dx\right)dt = r\sum_{n=M}^\infty \int_0^u \left(\frac{t}{t^2 + \frac{r^2\sigma_M^2}{\lambda_n^2}}\right)dt \\
&= \frac{r}{2}\sum_{n=M}^\infty \log\left(1 + \frac{u^2\lambda_n^2}{r^2\sigma_M^2}\right), \tag{60}
\end{aligned}
$$

where we have used the integral identity $\int_0^\infty \sin(tx)e^{-zx}dx = t/(t^2 + z^2)$ in the fourth line, which can be shown by integration by parts.

Using the properties of slowly varying functions, for any γ' and γ'' for which $\gamma' > \gamma > \gamma''$ one can find constants α_1, α_2 such that $\alpha_2 n^{-\gamma''} \gtrsim \lambda_n \gtrsim \alpha_1 n^{-\gamma'}$, that is, $\limsup \lambda_n n^{\gamma''}/\alpha_2 \le 1$ and $\liminf \lambda_n n^{\gamma'}/\alpha_1 \ge 1$. Since (60) is increasing is u, on the interval $\kappa_{3,M}^{-1} < u < U_M$, as $M \to \infty$,

$$
\begin{aligned}
A_M(u) \ge A_M(\kappa_{3,M}^{-1}) &= \frac{r}{2}\sum_{n=M}^\infty \log\left(1 + \frac{\lambda_n^2}{r^2\kappa_{3,M}^2\sigma_M^2}\right) \\
&\gtrsim \frac{r}{2}\int_M^\infty \log\left(1 + \frac{\alpha_1^2 y^{-2\gamma'}}{r^2\kappa_{3,M}^2\sigma_M^2}\right)dy \tag{61} \\
&= \frac{r}{2}\int_M^\infty \log\left(1 + (\beta_M y)^{-2\gamma'}\right)dy \tag{62}
\end{aligned}
$$

where

$$\beta_M = \left(\frac{\alpha_1^2}{r^2\kappa_{3,M}^2\sigma_M^2}\right)^{-1/2\gamma'}.$$

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Making the change of variables $w = \beta_M y$, the integral (62) becomes

$$\frac{r}{2\beta_M} \int_{M\beta_M}^{\infty} \log\left(1 + w^{-2\gamma'}\right) dw$$

Equations (24) and (26) together with the choice of γ'' imply $\kappa_{3,M}^2 \sigma_M^2 \lesssim CM^{-2\gamma''}$ for a constant $C > 0$, hence

$$\beta_M \lesssim C'M^{-\gamma''/\gamma'}$$

for another constant C'. Thus, we have shown that as $M \to \infty$,

$$A_M(\kappa_{3,M}^{-1}) \gtrsim \frac{rC'}{2} M^{\gamma''/\gamma'} \int_{C'M^{1-\gamma''/\gamma'}}^{\infty} \log(1 + w^{-2\gamma'}) dw.$$

Since $\log(1 + x) \sim x$ as $x \to 0$, we have

$$\begin{aligned}
A_M(\kappa_{3,M}^{-1}) &\gtrsim \frac{rC'}{2} M^{\gamma''/\gamma'} \int_{C'M^{1-\gamma''/\gamma'}}^{\infty} w^{-2\gamma'} dw \\
&= C''M^{\gamma''/\gamma'} M^{(1-\gamma''/\gamma')(1-2\gamma')} \\
&= C''M^{1-2(\gamma'-\gamma'')}
\end{aligned}$$

(63)

for another constant C''. Notice (63) tends to infinity so long as $\gamma' - \gamma''$ is chosen to be smaller than $1/2$. Now, returning to $J_{2,M}$, (57), (59), (63), and (26) imply

$$\begin{aligned}
|J_{2,M}| &\leq \int_{\kappa_{3,M}^{-1}}^{U_M} \exp(-A_M(u)) \frac{du}{u} \\
&\leq \exp(-A_M(\kappa_{3,M}^{-1})) \int_{\kappa_{3,M}^{-1}}^{U_M} \frac{du}{u} \\
&= \exp(-A_M(\kappa_{3,M}^{-1})) \log\left(\frac{\epsilon \kappa_{3,M}}{\kappa_{N+1,M}}\right) \\
&\lesssim C''' \exp\left(-C''M^{1-2(\gamma'-\gamma'')}\right) \log M, \quad \text{for some } C''' > 0 \\
&= o(M^{-(N-1)/2}).
\end{aligned}$$

Estimate for $J_{M,3}$:

For $J_{M,3}$, we have

$$|J_{M,3}| \leq \left| \int_{\kappa_{3,M}^{-1}}^{U_M} \exp\left(-\frac{u^2}{2}\right) \left(1 + \sum_{\eta(N)} \left(\prod_{m=3}^{N} \frac{1}{k_m!} \left(\frac{(iu)^m}{m!} \kappa_{m,M}\right)^{k_m}\right)\right) \frac{du}{u} \right|.$$

By bounding all the coefficients of the polynomial in u by their maximum value, we have

$$\begin{aligned}
|J_{M,3}| &\leq \max_{\eta(N)}\left(1, \prod_{m=3}^{N} \kappa_{m,M}^{k_m}\right) \int_{\kappa_{3,M}^{-1}}^{\infty} \exp\left(-\frac{u^2}{2}\right) \left(1 + \sum_{\eta(N)} \left(\prod_{m=3}^{N} \frac{1}{k_m!} \left(\frac{u^m}{m!}\right)^{k_m}\right)\right) du \\
&\sim \max_{\eta(N)}\left(1, \prod_{m=3}^{N} \kappa_{m,M}^{k_m}\right) \int_{\ell(M)^{-1}M^{1/2}}^{\infty} \exp\left(-\frac{u^2}{2}\right) p(u) du
\end{aligned}$$

(64)

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where $p(u)$ is a polynomial in u whose coefficients do not depend on M. Choosing a constant $C > 0$ large enough such that $p(u) \le Ce^u$ for all $u > \ell(M)^{-1}M^{1/2}$, we see

$$
\begin{aligned}
\int_{\ell(M)^{-1}M^{1/2}}^{\infty} \exp\left(-\frac{u^2}{2}\right) p(u) du &\le \int_{\ell(M)^{-1}M^{1/2}}^{\infty} \exp\left(-\frac{u^2}{2}\right) Ce^u du \\
&= Ce^{1/2} \int_{\ell(M)^{-1}M^{1/2}-1}^{\infty} e^{-w^2/2} dw \qquad (65) \\
&= C\sqrt{\frac{e\pi}{2}} \mathrm{Erfc}\left(\frac{\ell(M)^{-1}M^{1/2}-1}{\sqrt{2}}\right) \qquad (66)
\end{aligned}
$$

where $\mathrm{Erfc}(u) = \frac{2}{\sqrt{\pi}}\int_u^{\infty} e^{-w^2} dw$. Using the fact that $\mathrm{Erfc}(u) \sim e^{-u^2}/(\sqrt{\pi}u)$, ([11], equation 40:9:2) (64) and (66) imply

$$
\begin{aligned}
|J_{M,3}| &\le \max_{\eta(N)}\left(1, \prod_{m=3}^{N} \kappa_{m,M}^{k_m}\right) C\sqrt{\frac{e\pi}{2}} \mathrm{Erfc}\left(\frac{\ell(M)^{-1}M^{1/2}-1}{\sqrt{2}}\right) \\
&\sim O(1)\frac{\exp\left(-\frac{1}{2}\ell(M)^{-2}M\right)}{\ell(M)^{-2}M} \\
&= o(M^{-(N-1)/2})
\end{aligned}
$$

Combining the estimates for $J_{M,i}$, $i = 1, 2, 3$, together with Lemma 5.1 implies the desired result.
∎

6 A Numerical example

In this section, we will demonstrate the utility of the Edgeworth expansion given in Theorem 5.2 for computing the CDF of random variable of the form (1). Consider the example where $r = 1/2$, i.e. η_n is chi-squared with 1 degree of freedom, and the λ_n's are given simply by

$$\lambda_n = Cn^{-3/4}$$

where the normalization constant is $C = (2\zeta(6/4))^{-1/2} = (2\sum_{n=1}^{\infty} n^{-6/4})^{-1/2} \approx 0.4375$, where ζ denotes the Riemann zeta function. To compute the CDF of $Z = \sum_{n=1}^{\infty} \lambda_n(\eta_n - 1) = X_M + Y_M$, where X_M and Y_M are defined in (6), we will proceed in three steps.

1. Choose a truncation level $M \ge 1$. We will see below that M does not need to be too large. Once an M is chosen, one must be able to compute the CDF $F_{X_M}(x)$ of X_M, which is a finite sum of weighted chi-squared random variables. There are multiple techniques for doing this, for instance methods based on Laplace transform inversion, [20], [6], or Fourier transform inversion, [1].

2. Choose an $N \ge 3$ and compute the appropriate terms in Edgeworth expansion for the CDF $F_{Y_M}(x)$ of Y_M. For this example, σ_M defined in (7), and the $\kappa_{k,M}$'s defined in (25), can be computed in terms of the Riemann Zeta function:

$$
\begin{aligned}
\sigma_M^2 &= 2\sum_{n=M}^{\infty} \lambda_n^2 = 2C^2\left(\zeta(2\gamma) - \sum_{n=1}^{M-1} n^{-6/4}\right) \\
\kappa_{k,M} &= 2^{k-1}(k-1)!\sigma_M^{-2}\sum_{n=M}^{\infty} \lambda_n^k = 2^{k-1}(k-1)!\sigma_M^{-k}C^k\left(\zeta(k\gamma) - \sum_{n=1}^{M-1} n^{-3k/4}\right)
\end{aligned}
$$

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3. The CDF of the sum $Z = X_M + Y_M$ is given by the convolution

$$F_Z(x) = \int_{-\infty}^{\infty} F_{X_M}(x - y) dF_{Y_M}(y). \tag{67}$$

We compute this integral numerically in MATLAB using standard techniques.

We have studied approximations of Y_M for various values of M and N. Figures 1 and 2 give a sense of how good these approximations are. We look at the Edgeworth approximations to the density of Y_M and see how these behave as both M and N grow. Figure 1 shows plots of the $N = 2, 3, 4, 5$ Edgeworth approximations to the density of $\tilde{Y}_M$ for $M = 2, 5, 10$ and 20.

An Edgeworth expansion with values of $N \geq 2$ involves corrections to the normal distribution. Increasing N improves on this correction. If the improvement is already negligible if one goes from $N = 2$ to $N = 3$, then the distribution is close to normal. This appears to be the case in Figure 1 for already small values of M ($M = 10$).

What happens at smaller values of M? We note that in Figure 1, that even for $M = 2$, there seems to be no change in the Edgeworth correction as N goes from 4 to 5. Hence it appears that for small values of M, a high level of accuracy is already reached by $N = 5$ as it is hard to distinguish the $N = 4$ and $N = 5$ curves.

For this reason, we will use $N = 5$ to approximate the CDF of the full distribution Z. Figure 2 shows the CDF computed using an $N = 5$ Edgeworth expansion for Y_M for various values of M. Since the resulting approximation is nearly independent of $M = 2, 5, 10, 20$, it is clear that the convergence of the Edgeworth expansions is fast for this example. The techniques developed here are used in [19] to obtain the numerical evaluation of the CDF and PDF of the Rosenblatt distribution.
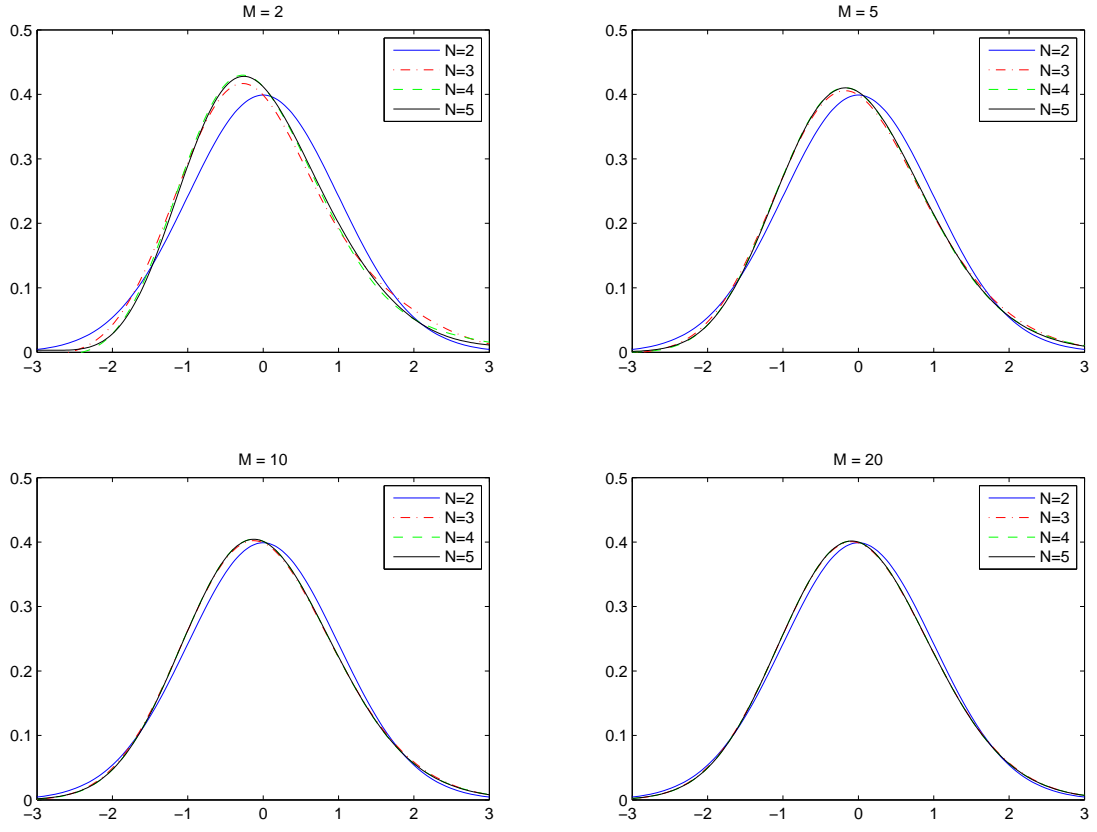
Figure 1: Edgeworth approximations to the density of $\tilde{Y}_M$ for various values of M. We can see that for this example, the converge of Y_M to a normal distribution is fast as M grows and increasing N beyond 5 causes a negligible change in the distribution function.
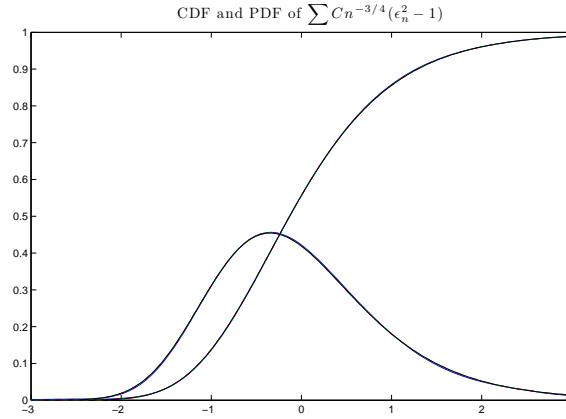
17

Figure 2: Approximation to the CDF and PDF of Z in the case $\lambda_n = Cn^{-3/4}$ using the $N = 5$ Edgeworth expansion for the tail Y_M. There are 4 curves corresponding to $M = 2, 5, 10$ and 20 in both curves and are almost indistinguishable suggesting fast convergence of this method.

References

[1] Joseph Abate and Ward Whitt. The Fourier-series method for inverting transforms of probability distributions. *Queueing Systems Theory Appl.*, 10(1-2):5–87, 1992.

[2] D. Applebaum. *Lévy Processes and Stochastic Calculus*. Cambridge University Press, Cambridge, UK, 2004.

[3] Søren Asmussen and Jan Rosiński. Approximations of small jumps of Lévy processes with a view towards simulation. *J. Appl. Probab.*, 38(2):482–493, 2001.

[4] J. Bertoin. *Lévy Processes*. Cambridge University Press, Cambridge, UK, 1996.

[5] R. N. Bhattacharya and R. Ranga Rao. *Normal approximation and asymptotic expansions*. John Wiley & Sons, New York-London-Sydney, 1976. Wiley Series in Probability and Mathematical Statistics.

[6] Antonia Castaño-Martínez and Fernando López-Blázquez. Distribution of a sum of weighted noncentral chi-square variables. *Test*, 14(2):397–415, 2005.

[7] R. L. Dobrushin and P. Major. Non-central limit theorems for nonlinear functionals of Gaussian fields. *Z. Wahrsch. Verw. Gebiete*, 50(1):27–52, 1979.

[8] A. Gut. *Probability: A Graduate Course*. Springer, New York, USA, 2005.

[9] Hui-Hsiung Kuo. *Introduction to stochastic integration*. Universitext. Springer, New York, 2006.

[10] U. Lorz and L. Heinrich. Normal and Poisson approximation of infinitely divisible distribution functions. *Statistics*, 22(4):627–649, 1991.

[11] Keith Oldham, Jan Myland, and Jerome Spanier. *An Atlas of Functions*. Springer, New York, second edition, 2009. With Equator, the atlas function calculator, With 1 CD-ROM (Windows).

[12] G. Peccati and M. S. Taqqu. *Wiener Chaos: Moments, Cumulants and Diagrams.* Bocconi Press and Springer Verlag, 2010. To appear.

[13] M. Rosenblatt. Independence and dependence. In *Proc. 4th Berkeley Sympos. Math. Statist. and Prob., Vol. II*, pages 431–443. Univ. California Press, Berkeley, Calif., 1961.

[14] Ken-iti Sato. *Lévy processes and infinitely divisible distributions*, volume 68 of *Cambridge Studies in Advanced Mathematics*. Cambridge University Press, Cambridge, 1999. Translated from the 1990 Japanese original, Revised by the author.

[15] I. G. Shevtsova. Sharpening the upper bound for the absolute constant in the Berry-Esseen inequality. *Teor. Veroyatn. Primen.*, 51(3):622–626, 2006.

[16] Fred W. Steutel and Klaas van Harn. *Infinite divisibility of probability distributions on the real line*, volume 259 of *Monographs and Textbooks in Pure and Applied Mathematics*. Marcel Dekker Inc., New York, 2004.

[17] M. S. Taqqu. The Rosenblatt process. In Richard Davis, Keh-Shin Lii, and Dimitris Politis, editors, *Selected Works of Murray Rosenblatt*. Springer Verlag, New York., 2011.

[18] M. S. Taqqu. Weak convergence to fractional Brownian motion and to the Rosenblatt process. *Z. Wahrscheinlichkeitstheorie und Verw. Gebiete*, 31:287–302, 1974/75.

[19] M. S. Veillette and Murad S. Taqqu. Properties and numerical evaluation of the Rosenblatt distribution. Preprint.

[20] M. S. Veillette and Murad S. Taqqu. Technique for computing the pdfs and cdfs of non-negative infinitely divisible random variables. Available at http://arxiv.org/abs/1005.2614.